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PICARD'S THEOREM AND BROWNIAN MOTION([1])

BY

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ABSTRACT. Properties of the paths of two dimensional Brownian motion are used as the basis of a proof of the little Picard theorem and its analog for complex valued functions, defined on simply connected n dimensional manifolds, which map certain diffusions into Brownian motion.

1. **Introduction.** Let C be the complex plane, f be a nonconstant entire function, and a and b be distinct complex numbers. It is possible to find a closed Brownian motion path in C which has an image under f that misses both a and b and is not homotopic to 0 in $C - \{a, b\}$. This can be used to show that either a or b is in the range of f, which proves Picard's little theorem. The proof given here follows this outline.

Brownian motion will always mean two dimensional Brownian motion in C, and Z_t, $0 \leq t < \infty$, will always be standard Brownian motion. Probability and expectation associated with Z_t given $P(Z_0 = a) = 1$ will be denoted by P_a and E_a. It is a theorem of P. Lévy that if f is a nonconstant entire function then $f(Z_t)$, $0 \leq t < \infty$, is also a Brownian motion, although perhaps moving with varying speed. More precisely, if $\eta(t) = \int_0^t |f'(Z_t)|^2\, dt$, then $\eta(t)$ is almost surely strictly increasing to infinity and the process W_t, $0 \leq t < \infty$, defined by $W_{\eta(t)} = f(Z_t)$ is a standard Brownian motion started at $f(Z_0)$. See McKean [5, p. 109] for a derivation. This reduces the study of the image paths $f(Z_t)$, $0 \leq t < \infty$, to the study of Brownian motion paths.

A diffusion on an n dimensional manifold will be called recurrent if, with probability 1, it returns to each neighborhood of its starting point at arbitrarily large times. It will be shown that a continuous complex valued function on a simply connected manifold which maps a recurrent diffusion into Brownian motion (in the sense of Lévy's theorem) has the Picard property, i.e., the range of f omits at most one complex number. (In §5 it will be shown how this condition can sometimes be expressed as a condition on the derivatives of f.) Thus to each recurrent diffusion corresponds a collection of functions with the Picard property, although sometimes, for example if the diffusion moves in only one dimension, it

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may be empty. If the diffusion is standard Brownian motion it will be seen that the associated functions are exactly those which are either entire or anti-entire $(\overline{f(z)}, f$ entire).

A theorem of Ito and McKean (essentially Theorem 3.1 here; see [5, p. 111]) about the entangling of Brownian motion in the twice punctured plane is important, in fact central, in what follows. Since the original proof uses the modular function and is not elementary, a new proof based on a law of large numbers is provided here. This proof gives an idea of the rate at which the entanglement progresses.

To be concise, statements and equations which hold only almost surely will not always be so qualified in what follows. Likewise the Strong Markov Property for Brownian motion, used again and again, will not be cited each time it is used.

Next a few well-known facts about Brownian motion will be collected in the form of propositions. Since the first two are easily proved using Lévy's theorem, proofs for them will be sketched. These proofs are new.

PROPOSITION 1.1. *$P_a(Z_t$ ever hits $b) = 0$ if $b \neq a$.*

PROOF. Under P_0, $(a-b)e^{Z_t} + b$ is a Brownian motion started at a. Clearly, it never hits b.

PROPOSITION 1.2. *Brownian motion is recurrent.*

PROOF. Using the Strong Markov Property and the fact that $aZ_t + b$ is also Brownian motion it is easy to reduce the proof to showing

$$P_r(|Z_t| \leq 1 \text{ for some } t > 0) = 1 \quad \text{if } r \text{ is real, } r > 1. \tag{1.1}$$

Now local properties of Brownian motion readily give

$$\lim_{r \to 1} P_r(|Z_t| \leq 1 \text{ for some } t > 0) = 1. \tag{1.2}$$

But

$$\begin{aligned} P_{r^{1/n}}(|Z_t| \leq 1 \text{ for some } t > 0) &= P_{r^{1/n}}(|Z_t^n| \leq 1 \text{ for some } t > 0) \\ &= P_r(|Z_t| \leq 1 \text{ for some } t > 0), \end{aligned}$$

the last equality holding since $f(z) = z^n$ is entire. Since $r^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$, (1.2) implies (1.1).

The next proposition, due to Kakutani, holds in much greater generality (see [3]). In the present case it follows from an extension of Lévy's theorem which deals with stopping times (see [1, p. 196]). Let R be the real line, and let $\Delta = \{z: |z| = ½\}$. Let $Q(r, z)$ and $P(e^{i\theta}/2, z)$ be the standard Poisson kernels on R and Δ respectively, that is, if $z = x + iy$,

$$Q(r, z) = \frac{1}{\pi} \frac{|y|}{y^2 + (x-r)^2} \quad \text{and} \quad P(e^{i\theta}/2, z) = \frac{1}{2\pi} \operatorname{Re} \frac{e^{i\theta/2} + z.}{e^{i\theta/2} - z}.$$

Then

PROPOSITION 1.3. (i) *If* $\mu = \inf\{t \geq 0: Z_t \in R\}$ *then the distribution of* Z_μ *under* P_z *is given by the measure* $Q(r, z)\,dr$.

(ii) *If* $\tau = \inf\{t \geq 0: Z_t \in \Delta\}$ *then, if* $|z| < ½$, *the distribution of* Z_τ *under* P_z *is given by the measure* $P(e^{i\theta}/2, z)\,d\theta$.

2. **Preliminaries.** Let R be the real numbers, I be the imaginary numbers, and $U = \{z: |z| \leq ½\}$. Let C^* be the space obtained by removing the points ± 1 from C and then identifying the points of U. If $h(t)$, $0 \leq t \leq T$, is a curve in C, that is a continuous function from $[0, T]$ to C, such that $h(t) \neq \pm 1$, $0 \leq t \leq T$, a condition that will be assumed to hold for all curves considered in this section, $h^*(t)$, $0 \leq t \leq T$, will be the curve in C^* which results from the projection of h.

If $h(0) \in U$, $h(T) \in U$, then h^* is a closed curve in C^*, and, since the fundamental group of the twice punctured plane is isomorphic to the free group with two generators, the entanglement of h^* in the points ± 1 can be represented by a word $w(h^*) = w$ consisting of the four letters a, a^{-1}, b, b^{-1}, where a and b stand for clockwise loops around 1 and -1 respectively and a^{-1} and b^{-1} for counterclockwise loops. It will always be assumed that w is written in the shortest possible way, so that $baa^{-1}b$ is disallowed, being replaced by bb. We divide the sixteen possible pairs of letters into three classes as follows. Class I: ab^{-1}, ba^{-1}, $a^{-1}b$, $b^{-1}a$. Class II: aa^{-1}, bb^{-1}, $a^{-1}a$, $b^{-1}b$. Class III: everything else. Let $i(w)$ be the initial letter of w, $l(w)$ be the last letter of w, and $|w|$ be the number of letters in w which are immediately followed by the inverse of the other letter, that is the number of times the Class I combinations occurs in w. For example, if $w = ab^{-1}aab$, $|w| = 2$. If h^* and g^* are two closed curves in C^* such that g^* begins where h^* ends we let h^*g^* stand for the curve h^* followed by g^* and we write $h^* \sim g^*$ if h^* is homotopic to g^*, $h^* \sim 0$ if h^* is homotopic to 0.

The first two of the following lemmas are immediate, and not proved.

LEMMA 2.1. *Let h be a curve in C starting and ending in U and let h_x, h_y, and h_{xy} be h reflected about the real, imaginary, and both axes respectively. Then* $|w(h^*)| = |w(h_x^*)| = |w(h_y^*)| = |w(h_{xy}^*)|$ *and the four words each start with a different letter and each end with a different letter.*

LEMMA 2.2. *Let h^* and g^* be two curves in C^* which begin and end in U, and let α, β and γ stand for $w(h^*)$, $w(g^*)$, and $w(h^*g^*)$ respectively. Then*

(i) $|\gamma| = |\alpha| + |\beta| + 1$ *if* $l(\alpha)i(\beta) \in$ *Class* I,

(ii) $|\gamma| \geq |\alpha| - |\beta| - 1$ *if* $l(\alpha)i(\beta) \in$ *Class* II,

(iii) $|\gamma| = |\alpha| + |\beta|$ *if* $l(\alpha)i(\beta) \in$ *Class* III.

LEMMA 2.3. *Let $h(t)$, $0 \leqslant t \leqslant T$, be a curve in C, such that $h(0) \in U$, $h(T) \in U$. Define $t_0 = 0$, and if $i \geqslant 1$ let*

$t_i = \inf\{t \geqslant t_{i-1}\colon |h(t)| = 1\}$, *if i is odd,*

$t_i = \inf\{t \geqslant t_{i-1}\colon h(t) \in R\}$, *if i is even and* $\operatorname{Re} h(t_{i-1}) \leqslant \operatorname{Im} h(t_{i-1})$,

$t_i = \inf\{t \geqslant t_{i-1}\colon h(t) \in I\}$, *if i is even and* $\operatorname{Re} h(t_{i-1}) > \operatorname{Im} h(t_{i-1})$.

Let $N = N(h) = \max\{k\colon t_k < T\}$. Then $|w(h^*)| \leqslant N$.

PROOF. For notational convenience let $T = t_{N+1}$. If $h(t_i) \notin U$ and $h(t_i) \notin (-\infty, -1) \cup (1, \infty)$, let l_{i+} be the shortest straight line from U to $h(t_i)$, oriented in that direction, and l_{i-} be this line oriented in the other direction. If $h(t_i) \in (-\infty, -1) \cup (1, \infty)$, let l_{i+} and l_{i-} be a straight line from $h(t_i)$ to U, hitting U on the same side of the real axis that $h(t_{i-1})$ is on, oriented as before. If $h(t_i) \in U$ let l_{i+} and l_{i-} be curves consisting of the single point $h(t_i)$. The curve $h(t)$, $t_{i-1} \leqslant t \leqslant t_i$, will be denoted by γ_i, and λ_i will stand for the curve $l_{(i-1)+}\gamma_i l_{i-}$. Now λ_i^* is closed, and furthermore $|w(\lambda_i^*)| = 0$. If i is even this is a consequence of the fact that λ_i stays entirely in one of the half planes $\operatorname{Re} z \geqslant 0$, $\operatorname{Re} z \leqslant 0$, $\operatorname{Im} z \geqslant 0$, or $\operatorname{Im} z \leqslant 0$, and if i is odd it holds since either λ_i stays entirely inside $\{z\colon |z| \leqslant 1\}$ (if $|h(t_{i-1})| \leqslant 1$), or else never hits $\{z \in R\colon -1 \leqslant z \leqslant 1\}$. Thus, since $h^* \sim \gamma_1^*\gamma_2^* \cdots \gamma_{N+1}^* \sim \lambda_1^*\lambda_2^* \cdots \lambda_{N+1}^*$, Lemma 2.2 implies $|w(h^*)| \leqslant \Sigma|w(\lambda_i^*)| + N = N$.

3. **Winding of Brownian motion.** If $\alpha \leqslant \gamma$ are two stopping times we will let $[Z_\alpha, Z_\gamma]$ stand for the curve Z_t, $\alpha \leqslant t \leqslant \gamma$, and P_0 and E_0 will be replaced by P and E for brevity. The σ-field generated by a collection φ of random variables is designated $\sigma(\varphi)$. Other notation is that introduced in §2 and Proposition 1.3.

LEMMA 3.1. *Let $\mathcal{F} = \sigma(Z_t, \tau \leqslant t < \infty)$. Given any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that*

(i) *$|P(A) - P_z(A)| \leqslant \epsilon P(A)$, if $A \in \mathcal{F}$ and $|z| \leqslant \delta$, and*

(ii) *$|E\phi - E_z\phi| \leqslant \in E\phi$, if ϕ is a nonnegative $\mathcal{F}$ measurable function and $|z| \leqslant \delta$.*

PROOF. By Proposition 1.3(ii),

$$P_z(A) = E_z P_w(A|Z_\tau = w) = \int_0^{2\pi} P_{e^{i\theta}/2}(A) P(e^{i\theta}/2, z)\, d\theta.$$

Since $P(e^{i\theta}/2, z) \longrightarrow P(e^{i\theta}/2, 0) = ½\pi$ uniformly as $z \longrightarrow 0$, (i) follows, and (ii) follows from (i) by first applying (i) to the sets $\{\phi > \lambda\}$ and then using the equalities $E\phi = \int_0^\infty P(\phi > \lambda)d\lambda$ and $E_z\phi = \int_0^\infty P_z(\phi > \lambda)d\lambda$.

Now let $r < ½$, define $\upsilon_r = \inf\{t \geqslant \tau\colon |Z_t| = r\}$, and let $\theta_r = [Z_\tau, Z_{\upsilon_r}]$.

LEMMA 3.2. $\operatorname{Limit}_{r\to 0} P(\theta_r^* \sim 0) = 0$.

PROOF. Let C' be $C - \{1\}$ with the points of U identified, and let θ_r' be the projection of θ_r onto C'. Then $\theta_r' \not\sim 0$ in C' implies $\theta_r^* \not\sim 0$ in C^*. Define $\mu_0 = 0$, and if $i > 1$ let

$$\mu_i = \inf\{t > \mu_{i-1}: Z_t \in (1, \infty)\}, \quad \text{if } i \text{ is odd},$$

$$\mu_i = \inf\{t > \mu_{i-1}: Z_t \in (-\infty, 1)\} \quad \text{if } i \text{ is even}.$$

Let $M - 1$ be the largest odd k such that $\mu_k < v_r$. Define $\eta_0, \eta_1, \ldots, \eta_M$ by $\eta_i = \mu_i$, $i < M$, $\eta_M = v_r$. Let $X_i(Z) = +1$ if $[Z_{\eta_{i-1}}, Z_{\eta_i}]$ makes a half loop clockwise around 1 ($\operatorname{Im} Z_{\eta_{i-}} > 0$ if i is even, <0 if i is odd), and -1 if the half loop is counterclockwise. Then $\Sigma_{i=1}^M X_i(Z)$ is twice the net number of winds θ_r' makes around 1, so $\theta_r' \sim 0$ in C' if and only if $\Sigma_{i=1}^M X_i(Z) = 0$. Now let $\rho_1, \rho_2, \ldots$ be any sequence of plus ones and minus ones, and define W_t, $0 \leqslant t \leqslant v_r$, by $W_t = Z_t$ if $t \in [\eta_{i-1}, \eta_i]$ and $\rho_i = +1$, $W_t = \bar{Z}_t$ if $t \in [\eta_{i-1}, \eta_i]$ and $\rho_i = -1$. Then the Strong Markov Property and the fact that reflected Brownian motion is still Brownian motion imply that W_t, $0 \leqslant t \leqslant v_r$, is also a Brownian motion and thus its distribution is exactly that of Z_t, $0 \leqslant t \leqslant v_r$. However $\Sigma_{i=1}^M X_i(W) = \Sigma_{i=1}^M \rho_i X_i(Z)$, so that, given $M = 2m_0$,

$$P(X_i(Z) = \rho_i, 1 \leqslant i \leqslant 2m_0) = P(X_i(Z) = 1, 1 \leqslant i \leqslant 2m_0),$$

implying

$$P\left(\sum_{i=1}^{M} X_i(Z) = 0 \mid M = 2m_0\right) = \binom{2m_0}{m_0} 2^{-2m_0}. \tag{3.1}$$

Now Proposition 1.1 guarantees $\lim_{r\to 0} v_r = \infty$, so that for any fixed k, $\lim_{r\to 0} P(v_r > \eta_k) = 1$, implying $P(\lim_{r\to 0} M = \infty) = 1$, which together with (3.1) gives $\lim_{r\to 0} P(\Sigma_{i=1}^M X_i(Z) = 0) = 0$, establishing the lemma.

Now let $y > 0$ be a fixed number satisfying both

$$y \leqslant \delta(.01) \quad \text{(see Lemma 3.1)}, \tag{3.2}$$

$$P(\theta_y^* \sim 0) \leqslant .01 \quad \text{(see Lemma 3.2)}, \tag{3.3}$$

and let θ^* and v stand for θ_y^* and v_y.

LEMMA 3.3. $E(|w(\theta^*)|^2) < \infty$.

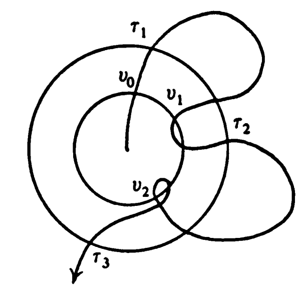
PROOF. Define $t_0, t_1, \ldots$ as in Lemma 2.3 and τ as in Proposition 1.3. On $\{\operatorname{Re} Z_{t_{2k-1}} \leqslant \operatorname{Im} Z_{t_{2k-1}}\}$,

$$P(|Z_{t_{2k}}| \leqslant y \mid Z_t, t \leqslant t_{2k-1}) = P_{Z_{t_{2k-1}}}(|Z_\tau| \leqslant y)$$

$$\geqslant \inf_{\{z: |z|=1, \operatorname{Re} z \leqslant \operatorname{Im} z\}} \int_{-y}^{y} P(r, z)\, dr = \alpha > 0,$$

by Proposition 1.3, with the same lower bound on $\{\operatorname{Im} Z_{t_{2k-1}} < \operatorname{Re} Z_{2k-1}\}$, using the analog of Proposition 1.3, (i), for the imaginary axis. Thus $P(t_{2k} > v | t_{2k-1} < v) \leqslant (1-\alpha)$, implying $P(t_{2k} > v) \leqslant (1-\alpha)^k$, so, by Lemma 2.3, $P(|w(\theta^*)| > 2k) \leqslant (1-\alpha)^k$, and thus $E(|w(\theta^*)|^2) < \infty$.

Now define $v_0 = \inf\{t \geqslant 0\colon |Z_t| = y\}$, $\tau_i = \inf\{t \geqslant v_{i-1}\colon |Z_t| = ½\}$, $i \geqslant 1$, and $v_i = \inf\{t > \tau_i\colon |Z_t| = y\}$, $i \geqslant 1$. The following picture may make things a little clearer.



Note that if the path is outside the circle $\{z\colon |z| \leqslant ½\}$ then $\tau_i < t < v_i$ for some $i \geqslant 1$. Define $d_i = [Z_{\tau_i}, Z_{v_i}]$, $S_n = [Z_0, Z_{v_n}]$, $H_n = [Z_{\tau_{2n-1}}, Z_{v_{2n}}]$, and $F_n = \sigma(Z_t, 0 \leqslant t \leqslant v_n)$. Note that $d_1^* \sim \theta^*$, $S_n^* \sim d_1^* d_2^* \cdots d_n^*$, $S_{2n}^* \sim H_1^* \cdots H_n^*$, and $H_k^* \sim d_{2k-1}^* d_{2k}^*$. Furthermore, under P, $w(d_1^*)$, $w(d_2^*)$, ... are identically distributed, as are $w(H_1^*)$, $w(H_2^*)$, ..., since Z_{τ_n} is uniformly distributed on $\{z\colon |z| = ½\}$.

Lemma 3.4. $P(|w(H_1^*)| \geqslant 1) > 1/5$.

Proof. By Lemma 2.2, $P(|w(H_1^*)| \geqslant 1) \geqslant P(l(w(d_1^*))i(w(d_2^*)) \in \text{Class I})$. Since, under P, Z_t reflected about any line through 0 has the same distribution as Z_t, Lemma 2.1 and (3.3) give

$$P(l(w(d_1^*)) = a) = P(d_1^* \not\sim 0)/4 \geqslant .99/4, \quad \text{and}$$

$$P(i(w(d_1^*)) = b^{-1}) = P(d_1^* \not\sim 0)/4 \geqslant .99/4,$$

so, since $|Z_{v_1}| \leqslant y$,

$$P(i(w(d_2^*)) = b^{-1} | F_1) = P_{Z_{v_1}}(i(w(d_2^*)) = b^{-1}),$$

$$\geqslant .99P(i(w(d_2^*)) = b^{-1}) \geqslant (.99)^2/4,$$

the last inequality using (3.3). Thus $P(i(w(d_2^*)) = b^{-1} | l(w(d_1^*)) = a) \geqslant .99^2/4$, so that $P(i(w(d_2^*)) = b^{-1}$ and $l(w(d_1^*)) = a) \geqslant (.99)^3/16$. Since the same bound holds for the other three pairs in Class I, we get $P(|w(H_1^*)| \geqslant 1) \geqslant 4((.99)^3/16) > 1/5$.

Finally, let $y_n = |w(S_{2n}^*)| - |w(S_{2n-2}^*)|$. Since $S_{2n-2}^* H_n^* \sim S_{2n}^*$, and since $E|w(d_i^*)|^2 = E|w(\theta^*)|^2 < \infty$, Lemma 2.2 gives

$$Ey_n^2 \leqslant E[(|w(H_n^*)| + 1)^2] \leqslant E[(|w(d_{2n-1}^*)| + |w(d_{2n}^*)| + 2)^2] = \beta < \infty.$$

Now let $A_n^{-1} = \{i(H_n^*)) = a^{-1}\}$ and similarly define B_n^{-1}, B_n, and A_n, and let I denote the indicator function. Then, by Lemmas 2.2, 3.1, 2.1, and 3.4, on $\{l(w(S_{2n-2}^*)) = a\}$ we have

$$\begin{aligned} E(y_n|F_{2n-2}) &\geqslant E[|w(H_n^*)|I(A_n \cup B_n) + (|w(H_n^*)| + 1)I(B_n^{-1}) \\ &\qquad - (|w(H_n^*)| + 1)\ I(A_n^{-1})|F_{2n-2}] \\ &= E_{Z_{v_{2n-2}}}[|w(H_1^*)|I(A_1 \cup B_1) + (|w(H_1^*)| + 1)I(B_1^{-1}) \\ &\qquad - (|w(H_1^*) + 1)\ I(A_1^{-1})] \\ &\geqslant .99E[|w(H_1^*)|I(A_1 \cup B_1 \cup B_1^{-1}) + I(B_1^{-1})] \\ &\quad - 1.01E[|w(H_1^*)|I(A_1^{-1}) + I(A_1^{-1})] \\ &= .99[3E|w(H_1^*)|/4 + P(H_1^* \not\sim 0)/4] - 1.01[E|w(H_1^*)|/4 - P(H_1^* \not\sim 0)/4] \\ &= .49E|w(H_1^*)| - .005P(H_1^* \not\sim 0) \geqslant .49/5 - .005 \geqslant 1/20, \end{aligned}$$

with the same holding on $\{l(w(S_{2n-2}^*)) = b, b^{-1}$, or $a^{-1}\}$. Furthermore, on $\{S_{2n-2}^* \sim 0\}$,

$$\begin{aligned} E(y_n|F_{2n-2}) &= E(|w(H_n^*)||\ F_{2n-2}) \\ &= E_{Z_{v_{2n-2}}}|w(H_1^*)| \geqslant .99E|w(H_1^*)| \geqslant .99/5 > 1/20. \end{aligned}$$

Thus, $E(y_n|F_{2n-2}) \geqslant 1/20$. Let $\Delta_n = E(y_n|F_{2n-2})$ and $a_n = y_n - \Delta_n$. Now $E(a_n^2) = E(E(a_n^2|F_{2n-2})) \leqslant EE(y_n^2|F_{2n-2}) = Ey_n^2 = \beta$. Thus, if $h_n = \Sigma_{i=1}^n a_i/i$, then $(h_n, F_{2n-2}, n \geqslant 1)$ is an L^2 bounded martingale, since $\|h_n\|_2^2 \leqslant \beta\Sigma_{i=1}^n 1/i^2$, and so converges (a.e.) [2, p. 319]. Kronecker's lemma implies $\lim_{n\to\infty} \Sigma_{i=1}^n a_i/n = 0$, so that $\liminf_{n\to\infty} \Sigma_{i=1}^n y_i/n \geqslant 1/20$, and $\lim_{n\to\infty} \Sigma_{i=1}^n y_i = \lim_{n\to\infty} |w(S_{2n}^*)| = \infty$.

If we define $y_i' = |w(S_{2i+1}^*)| - |w(S_{2i-1}^*)|, i \geqslant 1$, an exactly similar argument shows $\lim_{n\to\infty} \Sigma_{i=1}^n y_i' = \infty$, implying $\lim_{n\to\infty} |w(S_{2n+1}^*)| = \infty$. Thus,

$$\lim_{n\to\infty} |w(S_n^*)| = \infty. \tag{3.4}$$

Let $N = \sup\{k: |w(S_k^*)| = 0\}$, and let $E = \{N < \infty\}$. We have just proved $P(E) = 1$. Now by the argument used in the proof of Lemma 3.1, $P(E) = \int_0^{2\pi} (2\pi^{-1})P_{e^{i\theta}/2}(E)d\theta$, implying $P_{e^{i\theta}/2}(E) = 1$ for almost every θ, so that

$$P_z(E) = \int_0^{2\pi} P(e^{i\theta}/2, z)P_{e^{i\theta}/2}(E)d\theta = \int_0^{2\pi} P(e^{i\theta}/2, z)d\theta = 1, \qquad |z| < 1/2.$$

Now if both $|Z_0| \leqslant y$ and $|Z_{t_0}| \leqslant y$, there is a k_0 such that $v_{k_0} \leqslant t_0$

$\leqslant \tau_{k_0}$, and $[Z_0, Z_{t_0}]^* \sim S^*_{k_0}$. Since $S^*_k \not\sim 0$ if $k > N$, we may take $T = \tau_{N+1}$ in the following.

THEOREM 3.1. *Let* $|z| \leqslant y$. *There is a time* T, $P_z(T < \infty) = 1$, *such that* $t \geqslant T$ *and* $|Z_t| \leqslant y$ *implies* $[Z_0, Z_t]^* \not\sim 0$.

4. **Proof of Picard's little theorem.** Let g be an entire function and suppose that a and b are distinct complex numbers not in the range of g. Let $f(z) = (2g(z) - a - b)/(a - b)$. Then f is entire and ± 1 are not in the range of f. Let z_0 satisfy $|f(z_0)| \leqslant y/2$. That such a z_0 exists follows immediately from Proposition 1.2. and Lévy's theorem. For suppose $|f(0)| > y/2$, and let $\eta = \inf\{t > 0: |f(Z_t)| = y/2\}$. Then $P(\eta < \infty) = 1$. Let $r > 0$ be so small that $|z - z_0| \leqslant r$ implies $|f(z)| \leqslant y$. Let $\widetilde{C}$ be C with the points of $\{z: |z - z_0| \leqslant r\}$ identified. Then f induces in the natural way a continuous map from $\widetilde{C}$ to C^*, which will be called $\hat{f}$, and any curve which is homotopic to 0 in $\widetilde{C}$ is mapped by $\hat{f}$ into a curve which is homotopic to 0 in C^*, since this is a property of continuous maps from one topological space to another. A path will now be exhibited which is homotopic to 0 in $\widetilde{C}$ which has an image under $\hat{f}$ which is not homotopic to 0 in C^*, and this contradiction will prove Picard's little theorem.

Let W_t, $0 \leqslant t < \infty$, be a standard Brownian motion started at z_0. Then $f(W_t)$, $0 \leqslant t < \infty$, is a Brownian motion started at $f(z_0)$ and so, by Theorem 3.1, for almost every path W_t, $0 \leqslant t < \infty$, there is a time S such that $t \geqslant S$ and $|f(W_t)| \leqslant y$ implies $[f(W_0), f(W_t)]^* \not\sim 0$. Thus, for all $t \geqslant S$ satisfying $|W_t - W_0| \leqslant r$ we have $[f(W_0), f(W_t)]^* \not\sim 0$. Let λ be such a t, that is, let λ satisfy $\lambda \geqslant S$ and $|W_\lambda - W_0| = |W_\lambda - z_0| \leqslant r$. There $[f(W_0), f(W_\lambda)]^* \not\sim 0$. The existence of such a λ for almost every Brownian path is guaranteed by Proposition 1.2. Let γ stand for any path of the form W_t, $0 \leqslant t \leqslant \lambda$. Then the projection of γ into $\widetilde{C}$, which will be called $\widetilde{\gamma}$, is homotopic to 0 in $\widetilde{C}$ since $\widetilde{C}$ is simply connected. But the image of $\widetilde{\gamma}$ under $\hat{f}$ is $[f(W_0), f(W_\lambda)]^*$, which is not homotopic to 0, a contradiction.

5. **Extensions of Picard's theorem.** If D_t, $0 \leqslant t < \infty$, is a diffusion on a simply connected n-dimensional manifold Γ, $n \geqslant 2$, and if f is a continuous function from Γ to the complex numbers C, we will say that f maps D_t into Brownian motion if there is a continuous and almost surely strictly increasing random function $\eta_f(t) = \eta(t)$, with $\eta(0) = 0$, such that, if we define $B^f_t = B_t$ by $f(D_{\eta^{-1}(t)}) = B_t$, then B_t is standard Brownian motion up to the time $\lim_{t\to\infty} \eta(t) = \eta(\infty)$. As has been mentioned, analytic functions map Brownian motion Z_t into Brownian motion and it is very easy to check that here $\text{limit}_{t\to\infty}\, \eta_f(t) = \infty$. For general diffusions D_t note that on $\{\eta(\infty) < \infty\}$ we have, by the uniform continuity of Brownian motion paths, $\lim_{t\to\infty} f(D_t) = \lim_{t\to\eta(\infty)} B_t$ exists. Now the recurrence of D_t implies $\underline{\lim}_{t\to\infty} |f(D_t) - f(z_0)| = 0$, where z_0 is the starting point of D_t.

Thus, $\lim_{t\to\eta(\infty)} B_t = f(z_0)$ on $\{\eta(\infty) < \infty\}$, which with the aid of Proposition 1.1 proves $P(\eta(\infty) = \infty) = 1$. The proof of the following theorem now follows exactly in the manner of the proof of Picard's theorem.

THEOREM 5.1. *If D_t, $0 \leq t < \infty$, is a recurrent diffusion on a simply connected n-dimensional manifold Γ, $n \geq 2$, and if f is a continuous function which maps D_t into two dimensional Brownian motion, then the range of f omits at most one complex number.*

As mentioned, if the speed and drift of a diffusion D_t are known then Itô's lemma can be used to give conditions on the derivatives of f to ensure that $f(D_t)$ is Brownian motion. Suppose for simplicity that $\Gamma = R^2$ and that D_t is given by $dD_t = b(D_t)dt + \sigma(D_t)dW_t$, where $D_t = (D_t^1, D_t^2)$, $b(x) = (b_1(x), b_2(x))$, $\sigma(x) = \sigma_{ij}(x)$ is a 2×2 matrix, and $W_t = (W_t^1, W_t^2)$ is standard two dimensional Brownian motion. If $g = u + iv$ and if u and v have continuous second partial derivatives, then, if we let $u_i = \partial u/\partial x_i$, $u_{ij} = \partial^2 u/\partial x_i \partial x_j$, and $a_{ij} = \frac{1}{2}(\sigma_{i1}\sigma_{j1} + \sigma_{i2}\sigma_{j2})$, Itô's lemma gives

$$du(D_t) = \left(\sum_{i=1}^{2} u_i b_i + \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}u_{ij}\right)dt + \sum_{j=1}^{2}\left(\sum_{i=1}^{2} \sigma_{ij}u_i\right)dW_j$$

$$= p(u)dt + \gamma_1(u)dW_1 + \gamma_2(u)dW_2,$$

with a similar inequality for v.

An argument similar to one given in McKean [5, p. 109] gives that g maps D_t into two dimensional Brownian motion if it has no drift and infinitesimal increments which are orthogonal and of the same magnitude, that is if

(1) $p(u) = p(u) = 0$,

(2) $\gamma_1(u)\gamma_1(v) + \gamma_2(u)\gamma_2(v) = 0$,

(3) $\gamma_1^2(u) + \gamma_2^2(u) = \gamma_1^2(v) + \gamma_2^2(v)$.

Note that if D_t is standard Brownian motion, that is $b = 0$ and $\sigma_{ij} = 1$, if $i = j$, and 0 if $i \neq j$, then (1) becomes $\nabla^2 u = \nabla^2 v = 0$, and (2) and (3) together hold if and only if either the Cauchy-Riemann equations or their analog for anti-entire functions hold.

Now conditions on b and σ which guarantee that D_t is recurrent are known (see [4]), and using Theorem 5.1 and the above technique these can be translated directly to conditions on the derivatives of the real and imaginary parts of a complex valued function which guarantee the function has the Picard property.

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